Mechanics explained in seven pages

Excerpt from the `No bullshit guide to math and physics` by Ivan Savov

***Abstract*—Mechanics is the precise study of the motion of objects, the forces acting on them, and more abstract concepts such as momentum and energy. You probably have an intuitive understanding of these concepts already, but in the next seven pages you will learn how to use precise mathematical equations to support your intuition. All topics will be covered including prerequisites.**

INTRODUCTION

To solve a mechanics problem is to obtain the *equation of motion* $x(t)$ that describes the position of the object as a function of time. Once you know $x(t)$, you can answer any question pertaining to the motion of the object. To find the position of the object at $t = 3$[s], simply plug 3[s] into the equation of motion. To find the time(s) when the object reaches a distance of 20[m] from the origin, you must solve for t in $x(t) = 20$[m]. Many of the problems on your mechanics final exam will be of this form so, if you know how to find $x(t)$, you'll be in good shape to ace the final.

A. Dynamics is the study of forces

The first step toward finding $x(t)$ is to calculate all the *forces* that act on the object. Forces are the *cause* of motion, so if you want to understand motion you need to understand forces. Newton's second law $F = ma$ states that **a force acting on an object produces an acceleration inversely proportional to the mass of the object**. Once you have the acceleration, you can compute $x(t)$ using calculus. We will discuss the calculus procedure for getting from $a(t)$ to $x(t)$ shortly. For now, let's focus on the causes of motion: the forces acting on the object. There are many kinds of forces: the weight of an object $\vec{W}$ is a type of force, the force of friction $\vec{F}_f$ is another type of force, the tension in a rope $\vec{T}$ is yet another type of force and there are many others. Note the little arrow on top of each force, which is there to remind you that forces are *vector quantities*. Unlike regular numbers, forces act in a particular direction, so it is possible for the effects of one force to counteract the effects of another force. For example the weight of a flower pot is exactly counter-acted by the tension in the rope on which it is suspended, thus, while there are two forces acting on the pot, there is no *net force* acting on it. Since there is no net force to cause motion and since the pot wasn't moving to begin with, it will just sit there motionless despite the fact that there are forces acting on it! To find the *net force* acting on the object you have to calculate the sum of all the forces acting on the object $\vec{F}_{\text{net}} \equiv \sum \vec{F}$. Once you have the net force, you can use the formula $\vec{a}(t) = \frac{\vec{F}_{\text{net}}}{m}$ to find the acceleration of the object.

B. Kinematics is the study of motion

If you know the acceleration of an object $a(t)$ as a function of time and its initial velocity $v_i = v(0)$, you can deduce the object's velocity function $v(t)$ at all later times. This is because the acceleration function $a(t)$ describes the change in the velocity of the object. If you know the object started with an initial velocity of $v_i \equiv v(0)$, the velocity at a later time $t = \tau$ is equal to v_i plus the "total velocity change" between $t = 0$ and $t = \tau$. The mathematical way of saying this is $v(\tau) = v_i + \int_0^\tau a(t)\, dt$. The symbol $\int \cdot \, dt$ is called an *integral* and is a fancy way of finding the total of some quantity over a given time period. To find the change in the velocity we calculate the total of $a(t)$ between $t = 0$ and $t = \tau$.

To understand what is going on, it may be useful to draw an analogy with a scenario you are more familiar with. Consider the function $\text{ba}(t)$ which represents your bank account balance at time t, and the function $\text{tr}(t)$ which corresponds to the transactions (deposits and withdraws) on your account. The function $\text{tr}(t)$ describes the change in the function $\text{ba}(t)$, the same way the function $a(t)$ describes the change in $v(t)$. Knowing the balance of your account at the beginning of the month, you can calculate the balance at the end of the month as follows: $\text{ba}(30) = \text{ba}(0) + \int_0^{30} \text{tr}(t)\, dt$.

If you know the initial position x_i and the velocity function $v(t)$ you can find the position function $x(t)$ by using integration again. We find the position at time $t = \tau$ by adding up all the velocity (change in the position) between $t = 0$ and $t = \tau$. The formula is $x(\tau) = x_i + \int_0^\tau v(t)\, dt$.

The entire procedure for predicting the motion of objects can be summarized as follows:

$$\frac{1}{m}\underbrace{\left(\sum \vec{F} = \vec{F}_{\text{net}}\right)}_{\text{dynamics}} = \underbrace{a(t) \quad \xrightarrow{v_i + \int dt} \quad v(t) \quad \xrightarrow{x_i + \int dt} \quad x(t)}_{\text{kinematics}}. \tag{1}$$

If you understand the above equation, then you understand mechanics. My goal for the next couple of pages is to introduce you to all the concepts that appear in this equation and the relationships between them.

C. Other stuff

Apart from dynamics and kinematics, we'll discuss several other topics.

Newton's second law can also be applied to the study of objects in rotation. Angular motion is described by the angle of rotation $\theta(t)$, the angular velocity $\omega(t)$ and the angular acceleration $\alpha(t)$. The causes of angular acceleration is angular force, which we call *torque* $\mathcal{T}$. Apart from the change to angular quantities, the principles behind circular motion are exactly the same as those for linear motion.

During a collision between two objects, there will be a sudden spike in the contact force between them which can be difficult to measure and quantify. It is therefore not possible to use Newton's law $F = ma$ to predict the accelerations that occur during collisions. In order to predict the motion of the objects after the collision we must use a *momentum* calculation. An object of mass m moving with velocity $\vec{v}$ has momentum $\vec{p} \equiv m\vec{v}$. The principle of conservation of momentum states that **the total amount of momentum before and after the collision is conserved**. Thus, if two objects with initial momenta $\vec{p}_{i1}$ and $\vec{p}_{i2}$ collide, the total momentum before the collision must be equal to the total momentum after the collision:

$$\sum \vec{p}_i = \sum \vec{p}_f \qquad \Rightarrow \qquad \vec{p}_{i1} + \vec{p}_{i2} = \vec{p}_{f1} + \vec{p}_{f2}.$$

Using this equation, it is possible to calculate the final momenta $\vec{p}_{f1}$, $\vec{p}_{f2}$ of the objects after the collision.

Another way of solving physics problems is to use the concept of energy. Instead of trying to describe the entire motion of the object, we can focus only on the initial parameters and the final parameters. The law of conservation of energy states **the total energy of the system is conserved**. Knowing the total initial energy of a system allows us to find the final energy, and from this calculate the final motion parameters.

We will discuss all of these topics in the remainder of the document.

D. The plan

Most people think of mechanics as a horrible chore inflicted upon them which requires complicated mathematical prerequisites. Instead, I propose a different way of thinking about mechanics, namely, as an opportunity to play LEGO with a bunch of cool scientific building blocks.

Of course, you must realize that simply reading this seven page tutorial cannot make a mechanics expert out of you. Mechanics expertise comes from solving exercises on your own. What we *can* do in seven pages is go over all the important concepts and describe the physics formulas which connect these concepts. The good news is that **there are only twenty equations that you need to know to understand mechanics**. Don't worry about prerequisites. The hardest math you'll have to do is solving a quadratic equation and we will cover everything you need to know about vectors and calculus in the next section. Flip the page to continue. $\Rightarrow$

I. PRELIMINARIES

In order to understand the equations of physics you need to be familiar with vector calculations and to know a bit about integrals. We'll introduce these concepts in the next two subsections.

A. Vectors

Forces, velocities, and accelerations are vector quantities. A vector $\vec{v}$ can be expressed in terms of its *components* or in terms of its *length* and its *direction*.

- x-axis: the x-axis is the horizontal axis in the coordinate system
- y-axis: the y-axis is *perpendicular* to the x-axis
- v_x: the *component* of $\vec{v}$ along the x-axis
- v_y: the *component* of $\vec{v}$ along the y-axis
- $\hat{\imath} \equiv (1,0), \hat{\jmath} \equiv (0,1)$: unit vectors in the x and y-directions
- $\|\vec{v}\|$: the length of the vector $\vec{v}$
- θ: the angle that $\vec{v}$ makes with the x-axis

Given the xy-coordinate system, we can denote a vector in three equivalent ways: $\vec{v} \equiv (v_x, v_y) \equiv v_x\hat{\imath} + v_y\hat{\jmath} \equiv \|\vec{v}\|\angle\theta$.

Given a vector expressed as a length and direction $\|\vec{v}\|\angle\theta$, we calculate its components using the following formulas:

$$v_x = \|\vec{v}\|\cos\theta \qquad \text{and} \qquad v_y = \|\vec{v}\|\sin\theta.$$

Alternately, a vector expressed in component form $\vec{v} = (v_x, v_y)$ can be converted to the length-and-direction form as follows:

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right).$$

It is important that you know how to convert between these two forms; the component form is useful for calculations, whereas the length-and-direction form describes the geometry of vectors.

The *dot product* between two vectors $\vec{v}$ and $\vec{w}$ can be computed in two different ways:

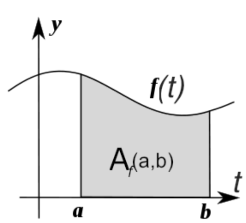
$$\vec{v}\cdot\vec{w} = v_x w_x + v_y w_y = \|\vec{v}\|\|\vec{w}\|\cos\phi,$$

where ϕ is the angle between the vectors $\vec{v}$ and $\vec{w}$. The dot product calculates how similar the two vectors are. For example, we have $\hat{\imath}\cdot\hat{\jmath} = 0$ since the vectors $\hat{\imath}$ and $\hat{\jmath}$ are orthogonal—they point in completely different directions.

B. Integrals

The integral of $f(t)$ corresponds to the computation of the *area under the graph* of $f(t)$ between two points:

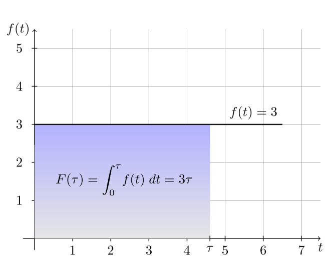
$$A_f(a,b) \equiv \int_a^b f(t)\,dt.$$



The symbol $\int$ is a mnemonic for *sum*, since the area under the graph corresponds in some sense to the sum of the values of the function $f(t)$ between $t=a$ and $t=b$. The integral is the total amount of f between a and b.

Consider for example the constant function $f(t) = 3$. Let's find the expression $F(\tau) \equiv A_f(0,\tau)$ that corresponds to the area under $f(t)$ between $t=0$ and the time $t=\tau$. We can easily find this area because the region under the graph is rectangular:

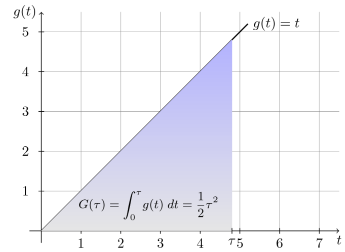
$$F(\tau) \equiv A_f(0,\tau) = \int_0^\tau f(t)\,dt = 3\tau.$$



The area of a rectangle is its height ($f(\tau) = 3$) times its width (τ).

Another important calculation is the area under the function $g(t) = t$. Let's compute $G(\tau) \equiv A_g(0,\tau)$, which is the area under the graph $g(t)$ between 0 and τ. This area is easily computed since the region under the graph is triangular:

$$G(\tau) \equiv A_g(0,\tau) = \int_0^\tau g(t)\,dt = \tfrac{1}{2}\tau^2.$$



The area of a triangle is the product of the length of the base (τ) times the height ($g(\tau) = \tau$) divided by two.

For the purpose of understanding mechanics, what you need to know is that the integral of a function is the total amount of the function accumulated during some time period. You should also try to remember the formulas:

$$\int_c^\tau a\,dt = a\tau + C, \qquad \int_c^\tau at\,dt = \tfrac{1}{2}a\tau^2 + C,$$

which correspond to the integral of a constant function and the integral of a line with slope a. Note that each time you give a general integral formula it will contain an additive constant term $+C$, which depends on the starting point of the area calculation. In the above examples we used $c=0$ as the starting point of the integral so the constant was zero $C=0$.

The integral of the sum of two functions is the sum of their integrals. Using this fact and the two formulas above, we can compute the integral for the function $f(t) = mt + b$ as follows:

$$\int_c^\tau (mt+b)\,dt = \tfrac{1}{2}m\tau^2 + b\tau + C. \tag{2}$$

Now that you know about vectors and integrals, we can start our discussion of the laws of physics.

II. KINEMATICS

Kinematics (from the Greek word for motion *kinema*) is the study of the trajectories of moving objects. The equations of kinematics can be used to calculate how long a ball thrown upward will stay in the air, or to calculate the acceleration needed to go from 0 to 100[km/h] in 5 seconds.

A. Concepts

The key notions used to describe the motion of an objects are

- t: the time. Time is measured in seconds [s].
- $x(t)$: the position of an object as a function of time—also known as the equation of motion
- $v(t)$: the velocity of the object as a function of time
- $a(t)$: the acceleration of the object as a function of time
- $x_i = x(0), v_i = v(0)$: initial position and velocity (initial conditions)

The position, velocity and acceleration functions ($x(t)$, $v(t)$, and $a(t)$) are connected. They all describe different aspects of the same motion. The function $x(t)$ is the main function since it describes the position of the object at all times. The velocity function describes the change in the position over time and it is measured in [m/s]. The acceleration function describes how the velocity changes over time and is measured in [m/s^2].

Assume now that we know the acceleration of the object $a(t)$ and that we want to find $v(t)$. The acceleration is the change in the velocity of the object. If we know that the object started with an initial velocity of $v_i \equiv v(0)$, and we want to find the velocity at later time $t=\tau$, we have to add up all the acceleration that the object felt during this time $v(\tau) = v_i + \int_0^\tau a(t)\,dt$. The velocity as a function of time is given by the initial velocity v_i plus the integral of the acceleration.

If we integrate the velocity function, we will obtain the position function $x(t)$. Thus, the procedure for finding $x(t)$ starting from $a(t)$ can be summarized as follows:

$$a(t) \quad \overset{v_i + \int dt}{\longrightarrow} \quad v(t) \quad \overset{x_i + \int dt}{\longrightarrow} \quad x(t).$$

We will now illustrate how to apply this procedure for the important special case of motion with constant acceleration.

B. Uniform acceleration motion

Suppose that an object starts from an initial position x_i with initial velocity v_i and undergoes a constant acceleration $a(t) = a$ from time $t = 0$ until $t = \tau$. What will be its velocity $v(\tau)$ and position $x(\tau)$ at time $t = \tau$?

We can find the velocity of the object by integrating the acceleration from $t = 0$ until $t = \tau$:

$$v(\tau) = v_i + \int_0^\tau a(t)\ dt = v_i + \int_0^\tau a\ dt = v_i + a\tau,$$

where we used the formula for the integral of a constant function. To obtain $x(t)$ we integrate $v(t)$ and obtain:

$$x(\tau) = x_i + \int_0^\tau v(t)\ dt = x_i + \int_0^\tau (v_i + at)\ dt = x_i + v_i\tau + \tfrac{1}{2}a\tau^2,$$

where we used the integral formula from equation (2). Note that the above integral calculations required the knowledge of the initial conditions x_i and v_i. This is because the integral calculations tell us the *change* in the quantities relative to their initial values.

We can summarize our findings regarding uniform acceleration motion (UAM) in the following three equations:

$$\begin{aligned} a(t) &= a, && \text{(by definition of UAM)} \\ v(t) &= at + v_i, && (3) \\ x(t) &= \tfrac{1}{2}at^2 + v_it + x_i. && (4) \end{aligned}$$

These equations fully describe all aspects of the motion of an object undergoing a constant acceleration $a(t) = a$ starting from $x(0) = x_i$ with initial velocity $v(0) = v_i$. There is also another very useful formula to remember:

$$v_f^2 = v_i^2 + 2a(x_f - x_i), \tag{5}$$

which is obtained by combining equation (3) and (4) in a particular way.

A special case of the above equations is the case with zero acceleration $a(t) = 0$. If there is no acceleration (change in velocity) then the velocity of the motion will be constant so we call this *uniform velocity motion* (UVM). The equations of motion for UVM are $v(t) = v_i$ and $x(t) = v_it + x_i$. If you understand the difference between UVM and UAM and the three formulas above, then you are ready to solve *any* kinematics problem.

C. Free fall

We say that an object is in *free fall* if the only force acting on it is the force of gravity. On the surface of the earth, the force of gravity will produce a constant acceleration of $a = -9.81$[m/s^2]. The negative sign is there because the gravitational acceleration is directed downward, and we assume that the y-axis points upward.

You can test your knowledge by trying the following practice problems.

0 to 100 in 5 seconds. You want to go from 0 to 100[km/h] in 5 seconds with your car. How much acceleration does your engine need to produce? Assume the acceleration is constant. Sol: Use (3). Ans: $a = 5.56$[m/s^2].

Moroccan example. Suppose your friend wants to send you a ball wrapped in aluminum foil from his balcony, which is located at a height of $x_i = 44.14$[m]. At $t = 0$[s] he throws the ball straight down with an initial velocity of $v_i = -10$[m/s]. How long does it take for the ball to hit the ground? Sol: Solve for t in (4) using $a = -9.81$. Ans: $t = 2.15$[s].

III. Projectile motion

We will now analyze an important kinematics problem in *two* dimensions. The motion of a projectile is described by:

- $\vec{r}(t) \equiv (x(t), y(t))$: the position (vector) of the object at time t
- $\vec{v}(t) \equiv (v_x(t), v_y(t))$: the velocity of the object as a function of time
- $\vec{a}(t) \equiv (a_x(t), a_y(t))$: the acceleration as a function of time

The motion of an object starts form an *initial* position an goes to a *final* position for which we'll use the following terminology:

- $t_i = 0$: initial time (the beginning of the motion)
- t_f: final time (when the motion stops)
- $\vec{v}_i = \vec{v}(0) = (v_x(0), v_y(0)) = (v_{ix}, v_{iy})$: the initial velocity at $t = 0$
- $\vec{r}_i = \vec{r}(0) = (x(0), y(0)) = (x_i, y_i)$: the initial position at $t = 0$
- $\vec{r}_f = \vec{r}(t_f) = (x(t_f), y(t_f)) = (x_f, y_f)$: the final position at $t = t_f$

Projectile motion is nothing more than two parallel one-dimensional kinematics problems: UVM in the x-direction and UAM in the y-direction.

A. Formulas

The acceleration felt by a flying projectile is

$$\vec{a}(t) = (a_x(t), a_y(t)) = (0, -9.81) \quad [\text{m/s}^2].$$

There is no acceleration in the x-direction (ignoring air friction) and we have a uniform downward acceleration due to gravity in the y-direction. Therefore, the equations of motion of the projectile are the following:

$$\begin{aligned} x(t) &= v_{ix}t + x_i, & y(t) &= \tfrac{1}{2}(-9.81)t^2 + v_{iy}t + y_i, \\ v_x(t) &= v_{ix}, & v_y(t) &= -9.81t + v_{iy}, \\ & & v_{yf}^2 &= v_{yi}^2 + 2(-9.81)(y_f - y_i). \end{aligned}$$

In the x-direction we have the equations of uniform velocity motion (UVM), while in the y-direction, we have equations of uniformly accelerated motion (UAM). Projectile motion problems can be decomposed into two separate sets of equations coupled through the time variable t.

Example. Let's now consider the example illustrated in Figure 1 which shows a rock being thrown with an initial velocity 8.96[m/s] at an angle of 51.3° from an initial height of 1[m]. You are asked to calculate the maximum height h that the rock will reach, and the distance d where it will fall back to the ground.

Your first step when reading any physics problem should be to extract the information from the problem statement. The initial position is $\vec{r}(0) = (x_i, y_i) = (0, 1)$[m]. The initial velocity is $\vec{v}_i = 8.96\angle 51.3°$[m/s], which is $\vec{v}_i = (8.96\cos 51.3°, 8.96\sin 51.3°) = (5.6, 7)$[m/s] in component form.

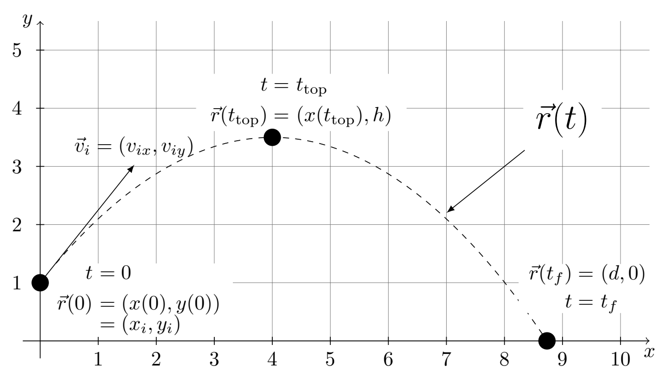


Figure 1. A rock is thrown with $\vec{v}_i = 8.96\angle 51.3°$[m/s] from $\vec{r}_i = (0, 1)$[m]. What is the maximum height h reached by the rock and the distance travelled d?

We can plug the values of $\vec{r}_i$ and $\vec{v}_i$ into the equations of motion and find the unknown quantities. When the object reaches its maximum height, it will have zero velocity in the y-direction: $v_y(t_{\text{top}}) = 0$. We can use this fact, and the $v_y(t)$ equation to find $t_{\text{top}} = 7/9.81 = 0.714$[s]. The maximum height is then obtained by evaluating the function $y(t)$ at $t = t_{\text{top}}$: $h = y(t_{\text{top}}) = 1 + 7(0.714) + \frac{1}{2}(-9.81)(0.714)^2 = 3.5$[m]. To find d, we must solve the quadratic equation $0 = y(t_f) = 1 + 7(t_f) + \frac{1}{2}(-9.81)(t_f)^2$ to find the time t_f when the rock will hit the ground. The solution is $t_f = 1.55$[s]. We then plug this value into the equation for $x(t)$ to obtain $d = x(t_f) = 0 + 5.6(1.55) = 8.68$[m]. We can verify that these answers match the trajectory illustrated in Figure 1.

IV. Dynamics

Dynamics is the study of the various forces that act on objects. Forces are vector quantities measured in Newtons [N]. In this section we will explore all the different kinds of forces.

A. Kinds of forces

Next we list all the forces which you are supposed to know about.

1) Gravitational force: The force of gravity exists between any two massive objects. The magnitude of the gravitational force between two objects of mass M[kg] and m[kg] separated by a distance r[m] is given by the formula $\vec{F}_g = \frac{GMm}{r^2}$[N], where $G = 6.67 \times 10^{-11} \left[\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}\right]$ is the *gravitational constant*.

On the surface of the earth, which has mass $M = 5.972 \times 10^{24}$[kg] and radius $r = 6.367 \times 10^6$[m], the force of gravity on an object of mass m is given by

$$F_g = \frac{GMm}{r^2} = \underbrace{\frac{GM}{r^2}}_{g} m = 9.81m = W. \tag{6}$$

We call this force the *weight* of the object. To be precise, we should write $\vec{W} = -mg\hat{\jmath}$ to indicate that the force acts *downward*—in the negative y-direction. Verify using your calculator that $\frac{GM}{r^2} \equiv g = 9.81$.

2) Force of a spring: A spring is a piece of metal twisted into a coil that has a certain natural length. The spring will resist any attempts to stretch it or compress it. The force exerted by a spring is given by

$$\vec{F}_s = -k\vec{x}, \tag{7}$$

where x is the amount by which the spring is displaced from its natural length and the constant k[N/m] is a measure of the *strength* of the spring. Note the negative sign: if you try to stretch the spring (positive x) then the force of a spring will pull against you (in the negative x-direction), if you try to compress the spring (negative x) it will push back against you (in the positive x-direction).

3) Normal force: The normal force is the force between two surfaces in contact. In this context the word *normal* means "perpendicular to the surface of." The reason why my coffee mug does not fall to the floor right now, is that the table exerts a normal force $\vec{N}$ on it keeping in place.

4) Force of friction: In addition to the normal force between surfaces, there is also the force of friction $\vec{F}_f$ which acts to prevent or slow down any sliding motion between the surfaces. There are two kinds of force of friction and both kinds of are proportional to the amount of normal force between the surfaces:

$$\max\{\vec{F}_{fs}\} = \mu_s\|\vec{N}\| \quad \text{(static)}, \qquad \vec{F}_{fk} = \mu_k\|\vec{N}\| \quad \text{(kinetic)}, \tag{8}$$

where μ_s and μ_k are the static and dynamic *friction coefficients*. Note that it makes intuitive sense that the force of friction should be proportional to the magnitude of the normal force $\|\vec{N}\|$: the harder the surfaces push against each other the more difficult it should be to make them slide. The equations in (8) make this intuition precise.

The static force of friction acts on objects that are not moving. It describes the *maximum* amount of friction that can exist between two objects. If a horizontal force greater than $F_{fs} = \mu_s N$ is applied to the object, then it will start to slip. The kinetic force of friction acts when two objects are sliding relative to each other. It always acts in the direction opposite to the motion.

5) Tension: A force can also be exerted on an object remotely by attaching a rope to the object. The force exerted on the object will be equal to the *tension* in the rope $\vec{T}$. Note that tension always pulls *away* from an object: you can't push a dog on a leash.

B. Force diagrams

Newton's 2nd law says that the *net* force on an object causes an acceleration:

$$\sum \vec{F} \equiv \vec{F}_{\text{net}} = m\vec{a}. \tag{9}$$

We will now learn how to calculate the net force acting on an object.

C. Recipe for solving force diagrams

1) Draw a diagram centred on the object. Draw the vectors of all the forces acting on the object: $\vec{W}$, $\vec{F}_s$, $\vec{N}$,$\vec{F}_{fs}$, $\vec{F}_{fk}$ and $\vec{T}$ as applicable.
2) Choose a coordinate system, and indicate clearly in the force diagram what you will call the positive x-direction, and what you will call the positive y-direction. All quantities in the subsequent equations will be expressed *with respect to* this coordinate system.
3) Write down this following "template":

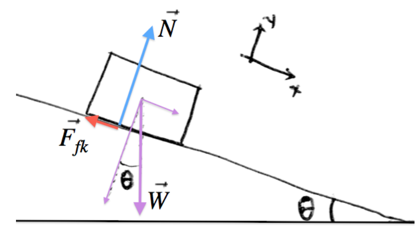
$$\sum F_x = \qquad\qquad = ma_x$$
$$\sum F_y = \qquad\qquad = ma_y$$

4) Fill in the template by calculating the x and y components of each of the forces acting on the object.
5) Solve the equations for the unknown quantities.

Let us now illustrate the procedure by solving an example problem.

Example. A block sliding down an incline with angle θ. The coefficient of friction between the block and the incline is μ_k. What is its acceleration?

Step 1: We draw a diagram which includes the weight, the normal force, and the kinetic force of friction.



Step 2: We choose the coordinate system to be tilted along the incline. This is important because this way the motion is purely in the x-direction, while the y-direction will be static.

Step 3,4: We copy over the empty template and fill in the components

$$\sum F_x = \|\vec{W}\|\sin\theta - F_{fk} = ma_x,$$
$$\sum F_y = N - \|\vec{W}\|\cos\theta \quad = 0.$$

Now substitute the known values to obtain

$$\sum F_x = mg\sin\theta - \mu_k N = ma_x,$$
$$\sum F_y = N - mg\cos\theta \quad = 0.$$

Step 5: We solve for a_x by first finding $N = mg\cos\theta$ in the y-equation and then substituting this value into the x-equation:

$$a_x = \frac{1}{m}\left(mg\sin\theta - \mu_k mg\cos\theta\right) = g\sin\theta - \mu_k g\cos\theta.$$

V. Momentum

During a collision between two objects there will be a sudden spike in the contact force between them that can be difficult to measure and quantify. It is therefore not possible to use Newton's law $F = ma$ to predict the accelerations that occur during collisions. We must use a *momentum* calculation to predict the motion of the objects after a collision.

A. Definition

The momentum of a moving object is equal to the velocity of the moving object multiplied by the object's mass:

$$\vec{p} = m\vec{v} \qquad [\mathrm{kg\,m/s}]. \tag{10}$$

Momentum is a vector quantity. If the velocity of the object is $\vec{v} = 20\hat{\imath} = (20, 0)$[m/s] and it has a mass of 100[kg] then its momentum is $\vec{p} = 2000\hat{\imath} = (2000, 0)$[kg m/s].

B. Conservation of momentum

The law of conservation of momentum states that the total amount of momentum before and after a collision is the same. In a collision involving two moving objects, if we know the initial momenta of the objects, we can calculate their momenta after the collision:

$$\sum \vec{p}_{\text{in}} = \sum \vec{p}_{\text{out}} \quad \Rightarrow \quad \vec{p}_{1,\text{in}} + \vec{p}_{2,\text{in}} = \vec{p}_{1,\text{out}} + \vec{p}_{2,\text{out}}. \tag{11}$$

This conservation law is one of the furthest reaching laws of physics you will learn in mechanics. The quantity of motion (momentum) cannot be created or destroyed, it can only be exchanged between systems. This law applies very generally: for fluids, for fields, and even for collisions involving atomic particles described by the laws of quantum mechanics.

Example. You throw a piece of rolled up carton from your balcony on a rainy day. The mass of the object is 0.4[g] and it is thrown horizontally with a speed of 10[m/s]. Shortly after it leaves your hand, the carton collides

with a rain drop of weight 2[g] falling straight down at a speed of 30[m/s]. What will be outgoing velocity of the objects if they stick together after the collision? Sol: The conservation of momentum equation says that: $\vec{p}_{in,1} + \vec{p}_{in,2} = \vec{p}_{\text{out}}$ so $0.4 \times (10, 0) \quad + \quad 2 \times (0, -30) \quad = \quad 2.4 \times \vec{v}_{\text{out}}$. Ans: $\vec{v}_{\text{out}} = (1.666, -25.0)$[m/s].

VI. Energy

Instead of finding the position function $x(t)$, we can solve physics problems using *energy* calculations. The key idea is the principle of *total energy conservation*, which tells us that, in any physical process, the sum of the initial energies is equal to the sum of the final energies.

A. Concepts

Energy is measured in Joules [J] and it arises in several different contexts:

- K — **Moving objects:** An object of mass m moving at velocity $\vec{v}$ has *kinetic energy* $K = \frac{1}{2}m\|\vec{v}\|^2$[J].
- W — **Moving objects by force:** If a constant force $\vec{F}$ acts on a object during a displacement $\vec{d}$, then the *work* done by this force is $W = \vec{F}\cdot\vec{d}$[J]. Positive work corresponds to energy being added to the system. Negative work corresponds to energy being removed from the system.
- U_g — **Gravitational potential energy:** The gravitational potential energy of an object raised to a height h above the ground is given by $U_g = mgh$[J], where m is the mass of the object and $g = 9.81$[m/s^2] is the gravitational acceleration on the Earth.
- U_s — **Spring potential energy:** The potential energy stored in a spring when it is displaced by x[m] from its relaxed position is given by $U_s = \frac{1}{2}k|x|^2$[J], where k[N/m] is the spring constant.

B. Conservation of energy

Consider a system which starts from an initial state (i), undergoes some motion and arrives at a final state (f). The law of conservation of energy states **energy cannot be created or destroyed in any physical process**. This means that the initial energy of the system plus the work that was *in*put into the system must equal the final energy of the system plus any work that the was *out*put:

$$\sum E_i \quad + W_{\text{in}} \quad = \quad \sum E_f \quad + W_{\text{out}}. \tag{12}$$

The expression $\sum E_{(a)}$ corresponds to the sum of the different types of energy the system has in state (a). If we write down the equation in full we have:

$$K_i + U_{gi} + U_{si} \quad + W_{\text{in}} \quad = \quad K_f + U_{gf} + U_{sf} \quad + W_{\text{out}}.$$

Usually, some of the terms in the above expression can be dropped. For example, we do not need to consider the spring potential energy U_s in physics problems that do not involve springs.

1) Work: The work done by a force $\vec{F}$ during a displacement $\vec{d}$ is:

$$W = \vec{F}\cdot\vec{d} = \|\vec{F}\|\|\vec{d}\|\cos\theta = \int_0^d \vec{F}(x)\cdot d\vec{x}.$$

Note the use of the dot product to obtain only the part of $\vec{F}$ that is pushing in the direction of the displacement $\vec{d}$. When the strength of the force changes during the motion, we must use integration to calculate the work.

2) Potential energy is stored work: Some types of work, like work against friction, are called *dissipative* since they just waste energy. Other kinds of work are called *conservative* since the work you do is not lost: it is converted into *potential energy*. The gravitational force and the spring force are conservative forces. Any work you do while lifting an object up into the air against the force of gravity is not lost but *stored* in the potential energy of the object. The gravitational potential energy of lifting an object from a height of $y = 0$ to a height of $y = h$ is defined as the *negative* of the work done:

$$U_g(h) \equiv -W_{\text{done}} = -\vec{F}_g\cdot\vec{h} = -(-mg\hat{\jmath})\cdot h\hat{\jmath} = mgh. \tag{13}$$

You can get *all* that energy back if you let go of the object. The energy will come back in the form of kinetic energy since the object will pick up speed during the fall.

The potential energy stored in a spring as it is compressed from $y = 0$ to $y = x$[m] is given by:

$$U_s(x) = -W_{\text{done}} = -\int_0^x \vec{F}_s(y)\cdot d\vec{y} = k\int_0^x y\,dy = \tfrac{1}{2}kx^2. \tag{14}$$

Example. An investment banker is dropped (from rest) from a 100[m]-tall building. What is his speed when he hits the ground? We'll use the formula $\sum E_i = \sum E_f$, where $_i$ corresponds to the top and $_f$ is at the bottom of the building. We have $K_i + U_i = K_f + U_f$ and after plugging-in the numbers we obtain $0 + m(9.81)(100) = \frac{1}{2}m(v_f)^2 + 0$. When we cancel the mass m from both sides of the equation, we're left with $9.81(100) = \frac{1}{2}(v_f)^2$, which we can solve for v_f. We find $v_f = \sqrt{2(9.81)(100)} = 44.29$[m/s]. This is like 160[km/h]. Ouch! That will de*f*initely hurt.

VII. Uniform circular motion

Circular motion is different from linear motion. In linear motion the acceleration leads to changes in the speed of the object whereas in circular motion the speed of the object stays constant, but the velocity continuously changes direction. We'll now discuss some techniques and concepts which are well suited for describing circular motion.

A. New coordinate system

Instead of the usual coordinate system $\hat{x}, \hat{y}$ which is static, we'll use a new coordinate system $\hat{t}, \hat{r}$ that is "attached" to the object:

- $\hat{r}$: the *radial* direction always points toward the centre of rotation
- $\hat{t}$: the *tangential* direction in the instantaneous direction of motion of the object. The name comes from the Greek word for "touch" (imagine a straight line "touching" the circle of rotation).

The $\hat{r}\hat{t}$-coordinate system makes it very easy to describe the velocity and the acceleration of an object undergoing circular motion. The velocity of the object is always in the tangential direction $\vec{v} = (0, v_t)_{\hat{r}\hat{t}} = 0\hat{r} + v_t\hat{t}$. The constant v_t is called the *tangential velocity*. The acceleration of a rotating object is always in the radial direction $\vec{a} = (a_r, 0)_{\hat{r}\hat{t}} = a_r\hat{r}$. The constant a_r is called the *radial acceleration*.

B. Radial acceleration

The defining feature of circular motion is the presence of an acceleration that acts towards the centre of rotation, perpendicularly to the direction of motion. The result of this inward acceleration is that the object will follow a circular path. The radial acceleration a_r of an object moving in a circle of radius R with a tangential velocity v_t is given by:

$$a_r = \frac{v_t^2}{R}. \tag{15}$$

Using Newton's second law $\vec{F} = m\vec{a}$, we can also calculate the magnitude of the radial force F_r which is the cause of this rotation: $F_r = ma_r = m\frac{v_t^2}{R}$. In words, this equation tells us the amount of radial force that is *required* in order to keep an object moving in a circular path.

Example. A rock of mass $m = 0.3$[kg] is swinging around in a horizontal circle attached at the end of a rope. The tangential velocity of the rock is $v_t = 50$[m/s] and the radius of the circle is $R = 1.5$[m]. What is the tension T in the rope? The tension in the rope $\vec{T}$ is the force that is causing the rotation. We have $F_r = T = ma_r$, from which we find that the tension in the rope must be $T = m\frac{v_t^2}{R} = 500$[N].

VIII. Angular motion

The basic concepts used to describe the rotation of objects are directly analogous to the concepts for linear motion: position, velocity, acceleration, force, momentum, and energy. This means that you already know all the equations of angular motion—you just have to do a "search and replace" with the new quantities. We'll now look into this more closely.

A. Concepts

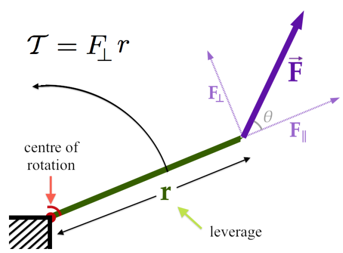
The new concepts used to describe the angular motion of objects are

- The angular kinematics quantities:
  - $\theta(t)$[rad]: the angular position
  - $\omega(t)$[rad/s]: the angular velocity
  - $\alpha(t)$[rad/s^2]: the angular acceleration
- I[kg m^2]: the *moment of inertia* of an object tells you how difficult it is to make it turn. The quantity I plays the same role in angular motion as the mass m plays in linear motion.
- $\mathcal{T}$[N m]: *torque* measures angular force, the cause of rotation
- $L = I\omega$[kgm^2/s]: the *angular momentum* of a rotating object describes the "quantity of rotational motion."
- $K_r = \frac{1}{2}I\omega^2$[J]: the *angular* or *rotational* kinetic energy quantifies the amount of energy an object has by virtue of its rotational motion.

B. Formulas

Instead of talking about position x, velocity v, and acceleration a, we'll now talk about the angular position θ, angular velocity ω, and angular acceleration α. Except for this change of ingredients, the *recipe* for fining the equation of motion remains the same: $\alpha(t) \overset{\omega_i+\int dt}{\longrightarrow} \omega(t) \overset{\theta_i+\int dt}{\longrightarrow} \theta(t)$. In particular, if you apply this recipe to the case of *uniformly accelerated angular motion* ($\alpha(t) = \alpha$), we obtain the equations $\omega(t) = \alpha t + \omega_i$ and $\theta(t) = \frac{1}{2}\alpha t^2 + \omega_i t + \theta_i$, which are analogous to equations (3) and (4).

1) Torque: Torque is angular force. In order to get an object to rotate you must exert a torque on it. Torque is measured in Newton metres [N m]. The torque produced by a force depends on how far from the centre of rotation it is applied:



$$\mathcal{T} = F_{\perp}\, r = \|\vec{F}\| \sin\theta\, r, \quad (16)$$

where r is how far from the centre of rotation the force acts. Note only the $F_{\perp}$ component produces torque.

2) Moment of inertia: The quantity I is defined as follows:

$$I = \{ \text{ how difficult it is to make an object turn } \}.$$

The moment of inertia depends on the *mass distribution* of the object:

$$I_{\text{disk}} = \frac{1}{2}mR^2, \quad I_{\text{ring}} = mR^2, \quad I_{\text{sphere}} = \frac{2}{5}mR^2.$$

The quantity I plays the same role in the equations of angular motion as the mass m plays in the equations of linear motion.

3) Torque causes angular acceleration: The angular analogue of Newton's second law is the equation

$$\mathcal{T} = I\alpha, \quad (17)$$

which indicates that the angular acceleration produced by the a toque $\mathcal{T}$ is inversely proportional to the object's moment of inertia.

4) Relation to linear quantities: We can relate the angular quantities θ, ω, and α to quantities like distance, velocity and acceleration by multiplying the angular quantity by the radius: $d = R\theta$, $v_t = R\omega$, $a_t = R\alpha$.

5) Angular momentum: The angular momentum of a spinning object measures the "amount of rotational motion" that the object has. The formula for the angular momentum of a an object with moment of inertia I rotating at an angular velocity ω is $L = I\omega$ [kg m^2/s]. The angular momentum of an object is a conserved quantity in the absence of torque $L_{\text{in}} = L_{\text{out}}$. This is analogous to the way momentum $\vec{p}$ is a conserved quantity in the absence of external forces.

6) Rotational kinetic energy: The kinetic energy of a rotating object is $K_r = \frac{1}{2}I\omega^2$[J]. This is the rotational analogue to the linear kinetic energy $\frac{1}{2}mv^2$. The amount of work produced by a torque $\mathcal{T}$ which is applied during an angular displacement of θ is $W = \mathcal{T}\theta$[J].

Example. A solid disk of mass 20[kg] and radius 30[cm] is spinning with an angular velocity of 20[rad/s]. A brake pad is applied to the edge of the disk producing $F_{fk} = 60$[N]. How long before the disk stops?

To solve this rotational kinematics problem we need to find the angular acceleration produced by the brake. We can do this using the equation $\mathcal{T} = I\alpha$. We must find $\mathcal{T}$ and I_{disk} and then solve for α. The torque produced by the brake is calculated using the force-times-leverage formula: $\mathcal{T} = F_{\perp}r = 60 \times 0.3 = 18$[N m]. The moment of inertia of a disk is given by $I_{\text{disk}} = \frac{1}{2}mR^2 = \frac{1}{2}(20)(0.3)^2 = 0.9$[kg m^2]. Thus we have $\alpha = -\frac{18}{0.9} = -20$[rad/s^2]. We can now use the UAM formula for the angular velocity $\omega(t) = \alpha t + \omega_i$ and solve for the time when the motion will stop: $0 = \alpha t + \omega_i$. The disk will come to a stop after $t = -\omega_i/\alpha = 1$[s].

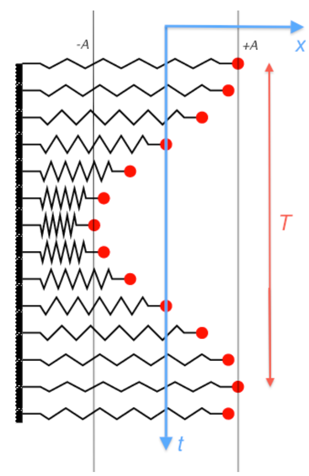
IX. Simple harmonic motion

We will now learn about *simple harmonic motion*, which describes oscillations and vibrations in mechanical systems.

A. Concepts

- A: The *amplitude* of the movement is how far the object moves back and forth relative to its centre position.
- $x(t)$[m], $v(t)$[m/s], $a(t)$[m/s^2]: position, velocity, and acceleration of the object as functions of time
- T[s]: the *period* of the object's motion. The period is how long it takes for the motion to repeat.
- f[Hz]: the *frequency* of the motion
- ω[rad/s]: *angular frequency*, $\omega = 2\pi f$.
- ϕ[rad]: the phase shift denoted by the Greek letter *phee*

The figure on the right illustrates a mass-spring system undergoing simple harmonic motion. Observe that the position of the mass as a function of time behaves like the cos function. From the diagram, we can identify two important parameters of the motion: the amplitude A, which describes the maximum displacement of the mass from the centre position and the period T, which describes how long it takes for the mass to come back to its initial position.



The equation which describes the position of the object as a function of time is the following:

$$x(t) = A\cos(\omega t + \phi). \quad (18)$$

The constant ω (omega) is called the *angular frequency* of the motion. It is related to the period T by the equation $\omega = \frac{2\pi}{T}$. The additive constant ϕ inside the cos is called the *phase constant* or *phase shift* and its value depends on the initial condition for the motion $x_i \equiv x(0)$.

1) Review of trigonometric functions: In order to understand the purpose of the three parameters A, ω, and ϕ, we'll now review the properties of the cos function. The function $f(t) = \cos(t)$ is a periodic function which oscillates between -1 and 1 with a period of 2π. As t goes from $t = 0$ to $t = 2\pi$ the function cos completes one full cycle.

To describe periodic motion with a different period, we can use a multiplier in front of the variable t inside the cos function: $f(t) = \cos(\omega t)$. If you want to have a periodic function with period T, you should use the multiplier constant $\omega = \frac{2\pi}{T}$. When you vary t from 0 to T, the function $\cos(\omega t)$ goes through one cycle because the quantity ωt goes from 0 to 2π. The *frequency* of a periodic motion describes how many times per second it repeats. The frequency is equal to the inverse of the period $f = \frac{1}{T} = \frac{\omega}{2\pi}$[Hz].

If we want to have oscillations that go between $-A$ and $+A$ instead of between -1 and 1, we can multiply the cos function by the appropriate *amplitude*: $f(t) = A\cos(\omega t)$.

The function $A\cos(\omega t)$ starts from its maximum value at $t = 0$. For a mass-spring system, this corresponds to the case when the motion begins with the spring maximally stretched $x_i \equiv x(0) = A$. In order to describe

other starting positions for the motion, it may be necessary to introduce a *phase shift* inside the cos function

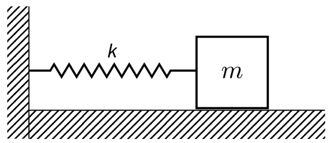
$$f(t) = A\cos(\omega t + \phi).$$

The constant ϕ must be chosen so that at $t = 0$, the function $f(t)$ correctly describes the initial position of the system. For example, the a harmonic motion which starts from the centre $x_i \equiv x(0) = 0$ is described by $x(t) = A\sin(\omega t) = A\cos\left(\omega t - \frac{\pi}{2}\right)$. Note that the function $x(t)$ correctly describes the initial condition $x_i \equiv x(0) = 0$.

B. Mass-spring system

Okay, enough math! It is time to learn about a physical system which exhibits simple harmonic motion: the mass-spring system. An object of mass m attached to a spring with spring constant k will undergo simple harmonic motion with angular frequency:

$$\omega = \sqrt{\frac{k}{m}}. \tag{19}$$



A stiff spring attached to a small mass will result in very rapid oscillations. A weak spring or a large mass will result in slow oscillations. A typical exam question will tell you k and m and ask about the period T. If you remember the definition of T, you can easily calculate the answer $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$[s].

We showed the position function $x(t)$ of the mass-spring system in equation (18). We can obtain the velocity and the acceleration functions of the mass-spring system by computing the derivatives of $x(t)$:

$$v(t) = -A\omega\sin(\omega t + \phi), \qquad a(t) = -A\omega^2\cos(\omega t + \phi). \tag{20}$$

1) Energy: Recall that the potential energy stored in a spring which is stretched (compressed) by a length x is given by the formula $U_s = \frac{1}{2}kx^2$. Since we know $x(t)$, we can obtain the potential energy of the mass-spring system as a function of time $U_s(t) = \frac{1}{2}k[x(t)]^2$. The kinetic energy of the mass as a function of time is given by $K(t) = \frac{1}{2}m[v(t)]^2$.

The potential energy reaches its maximum value $U_{s,\text{max}} = \frac{1}{2}kA^2$ when the spring is fully stretched or fully compressed. The kinetic energy is maximum when the mass passes through the center position. The maximum kinetic energy is given by $K_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}mA^2\omega^2$.

2) Conservation of energy: We know that, in the absence of dissipative forces, the energy of a system is conserved. The best way to understand SHM is to think of the energy in the system which shifts between the potential energy of the spring and the kinetic energy of the moving mass. When the spring is maximally stretched $x = \pm A$, the mass will have zero velocity and hence zero kinetic energy $K = 0$. At this moment all the energy of the system is stored in the spring $E_T = U_{s,\text{max}}$. The other important moment is when the mass has zero displacement but maximal velocity $x = 0, U_s = 0, v = \pm A\omega, E_T = K_{\text{max}}$, which corresponds to all the energy being stored as kinetic energy. The *total energy* of the system remains constant at all times: $E_T = U_s(t) + K(t) =$ constant.

You will often be asked in exercises to find the quantities $v_{\text{max}} = A\omega$ and $a_{\text{max}} = A\omega^2$. This is an easy task if you know the values of the amplitude A and the angular frequency ω.

Example. You are observing a mass-spring system build from a 1[kg] mass and a 250[N/m] spring. The amplitude of the oscillation is 10[cm]. Determine (a) the maximum speed of the mass, (b) the maximum acceleration, and (c) the total mechanical energy of the system.

First we find the angular frequency for this system $\omega = \sqrt{k/m} = \sqrt{250/1} = 15.81$[rad/s]. To find (a) we use the equation $v_{\text{max}} = \omega A = 15.81 \times 0.1 = 1.58$[m/s]. Similarly, we can find the maximum acceleration using $a_{\text{max}} = \omega^2 A = 15.81^2 \times 0.1 = 25$[m^2/s]. To solve (c), we can either calculate the total energy of the system by considering the potential energy of the spring when it is maximally extended (compressed) $E_T = U_s(A) = \frac{1}{2}kA^2 = 1.25$[J], or we can obtain the total energy from the maximum kinetic energy $E_T = K = \frac{1}{2}mv_{\text{max}}^2 = 1.25$[J].

Summary

The fundamental purpose of mechanics is to predict the motion of objects using equations. In the beginning of this tutorial, I made the claim that there are only twenty equations that you need to know in order to solve any physics problem. Let us now verify this claim and review the material.

Our goal was to find $x(t)$ for all times t. However, there are no equations of physics which will tell us $x(t)$ directly. Instead, we have Newton's second law (9), which tells us that the acceleration of the object $a(t)$ is equal to the *net force* acting on the object divided by the object's mass. To find $x(t)$ starting from $a(t)$, we must use integration (twice). The entire "mechanics recipe" is described by equation (1).

In the remainder of the document we applied this recipe in several different contexts. In Section II we looked at kinematics problems in one dimension, and derived equations (3) and (4), which describe the motion of a particle undergoing constant acceleration (UAM). In Section III we studied the problem of projectile motion by decomposing it into two separate kinematics subproblems: one in the x-direction (UVM) and one in the y-direction (UAM). In Section VII we studied the circular motion of objects and stated equation (15) which describes an important relationship between the radial acceleration, the tangential velocity and the radius of the circle of rotation. In Section VIII we described rotational motion in terms of angular quantities. We defined the concept of *torque* in equation (16) and used this concept to write down the angular equivalent of Newton's second law in equation (17). Finally, in Section IX we stated equation (18) which describes simple harmonic motion and formula (19), which is used to find the angular frequency of a mass-spring system.

We also discussed three conservation laws: the law of conservation of linear momentum $\sum \vec{p}_i = \sum \vec{p}_f$ from equation (11), the law of conservation of angular momentum ($L_i = L_f$), and the conservation of energy law in equation (12). Each of these three fundamental quantities is conserved overall and cannot be created nor destroyed. In Section V we described how momentum calculations can be used to analyze collisions, while in Section VI we used energy formulas like equations (13) and (14) to analyze the motion of objects in terms of energy principles.

As you can see, twenty equations really are enough for all of mechanics.

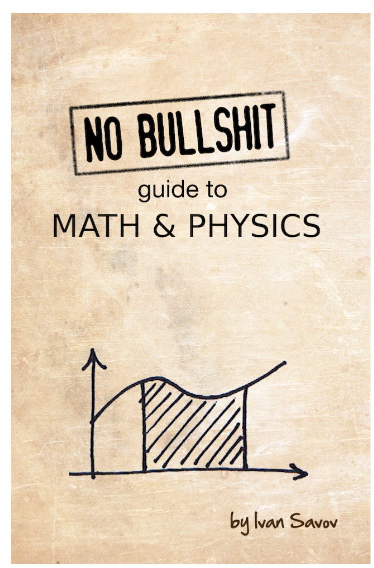
About the Book

If you want to learn more university-level math and physics, I invite you to check out my book, the **No bullshit guide to math and physics**.

This book contains short lessons on topics in math and physics, written in a style that is jargon-free and to the point. Often calculus and mechanics are taught as separate subjects. It shouldn't be like that. If you learn calculus without mechanics, it will be boring. If you learn mechanics without calculus, you won't truly understand. This textbook covers both subjects in an integrated manner and aims to highlight the connections between them.

Contents:

- HIGH SCHOOL MATH
- VECTORS
- **MECHANICS** (just 70 pages!)
- DIFFERENTIAL CALCULUS
- INTEGRAL CALCULUS



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For more information, see the book's website `minireference.com` or you can get in touch with me by email here `ivan.savov@gmail.com`.

You can also follow `@minireference` on twitter and check out the facebook page `fb.me/noBSguide`.

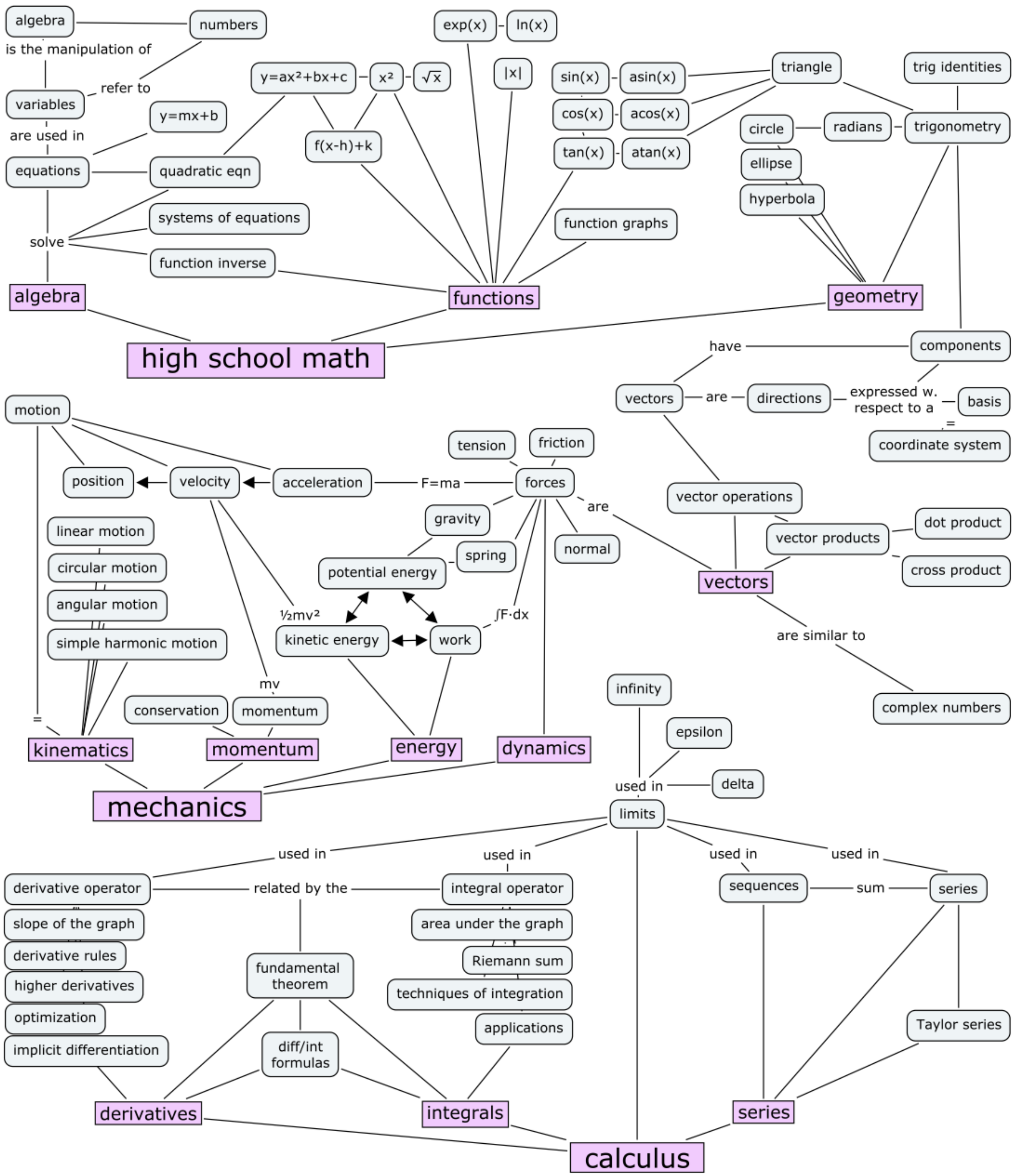

Figure 2. Each concept in this diagram corresponds to one section in the “No bullshit guide to math and physics,” by Ivan Savov, fourth edition, ISBN: 9780992001001.